NOTE ON MULTIPLY PERFECT NUMBERS.*

By Jacob Westlund.

In the Annals of Mathematics, ser. 2, vol. 2 (1900/01), p. 103, Dr. D. N. Lehmer proves that no multiply perfect numbers of multiplicity 3, containing less than three distinct primes, exist. In an earlier note on Multiply Perfect Numbers† all numbers of multiplicity 3 of the form $p_1^{a_1} p_2^{a_2} p_3$ were determined. The object of the present note is to determine all numbers of multiplicity 3 of the form $m = p_1^{a_1} p_2 p_3 p_4$ where p_1, p_2, p_3, p_4 are distinct primes and $p_1 < p_2 < p_3 < p_4$.

Defining a multiply perfect number as one which is an exact divisor of the sum of all its divisors, the quotient being the multiplicity, we have‡ in the present case

$$3 = \frac{p_1 - 1/p_1^{a_1}}{p_1 - 1} \prod_{i=2}^{4} \frac{p_i - 1/p_i}{p_i - 1} = \frac{p_1^{a_1+1} - 1}{p_1^{a_1}(p_1 - 1)} \prod_{i=2}^{4} \frac{p_i + 1}{p_i} \tag{1}$$

and

$$3 < \prod_{i=1}^{4} \frac{p_i}{p_i - 1} . \tag{2}$$

From the inequality (2) we infer that $p_1 = 2$ is the only possible value of p_1, since the maximum value of $\prod_{i=1}^{4} \frac{p_i}{p_i - 1}$ will exceed 3 only for $p_1 = 2$. Hence

$$\frac{3}{2} < \prod_{i=2}^{4} \frac{p_i}{p_i - 1} ,$$

which gives $p_2 < 7$, *i. e.* the only possible values of p_2 are 3 and 5.

I. Suppose $p_2 = 3$. Then we should have from (1)

$$3 = \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{4}{3} \cdot \frac{p_3 + 1}{p_3} \cdot \frac{p_4 + 1}{p_4}$$

or

$$9 = \frac{2^{a_1+1} - 1}{2^{a_1-2}} \cdot \frac{p_3 + 1}{p_3} \cdot \frac{p_4 + 1}{p_4} . \tag{3}$$

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* Read before the Chicago Section of the American Mathematical Society, January 3, 1902.

† Westlund, Annals of Mathematics, ser. 2, vol. 2 (1900/01), p. 172.

‡ *Cf.* Lehmer, *l. c.*

From this we infer that

$$\left(\frac{p_3+1}{p_3}\right)^2 > \frac{9\cdot 2^{a_1-2}}{2^{a_1+1}-1} > \frac{9}{8}$$

or

$$\frac{p_3+1}{p_3} > \frac{3}{4}\sqrt{2}\,.$$

Hence $p_3 < 17$, *i. e.* the only possible values of p_3 are 5, 7, 11, 13.

From (3) we see that $2^{a_1+1}-1$ must be divisible by p_4. Setting $2^{a_1+1}-1 = kp_4$ we have

$$\begin{aligned} 9\cdot 2^{a_1-2}p_3 &= k(p_3+1)(p_4+1) \\ &= (p_3+1)(2^{a_1+1}-1+k) \end{aligned} \tag{4}$$

which may be written in the following form

$$2^{a_1-2}(p_3-8) = (k-1)(p_3+1). \tag{5}$$

Hence we must have $k > 1$ and $p_3 > 8$, and the only possible values of p_3 are 11 and 13. It is easily seen that $p_3 = 13$ does not satisfy (5). For $p_3 = 11$ we get from (5)

$$3\cdot 2^{a_1-2} = (k-1)12$$

or

$$2^{a_1-4} = k-1,$$

and since $2^5\cdot 2^{a_1-4} = kp_4+1$, $k = \dfrac{33}{32-p_4}$. Hence $p_4 = 31$, $k = 33$, $a_1 = 9$. The corresponding number is $m = 2^9\cdot 3\cdot 11\cdot 31$ which by trial is found to be multiply perfect.

II. Suppose $p_2 = 5$. In this case we should have

$$3 = \frac{2^{a_1+1}-1}{2^{a_1}}\cdot\frac{6}{5}\cdot\frac{p_3+1}{p_3}\cdot\frac{p_4+1}{p_4} \tag{6}$$

which gives

$$\left(\frac{p_3+1}{p_3}\right)^2 > \frac{5\cdot 2^{a_1-1}}{2^{a_1+1}-1} > \frac{5}{4}$$

or

$$\frac{p_3+1}{p_3} > \frac{1}{2}\sqrt{5}.$$

Hence the only possible value of p_3 is 7. Setting, as in the first case, $2^{a_1+1} - 1 = kp_4$ we get from (6)

$$35 \cdot 2^{a_1-4} = k(p_4 + 1)$$

or

$$3 \cdot 2^{a_1-4} = k - 1,$$

and $k = \dfrac{35}{32 - 3p_4}$. Hence $p_4 < 11$ and since $p_4 > 7$ this is absurd. Therefore p_2 cannot be equal to 5, and the only multiply perfect number of multiplicity 3 of the form $m = p_1^{a_1} p_2 p_3 p_4$ is the number $2^9 \cdot 3 \cdot 11 \cdot 31$.

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